

$sl(2, \mathbb{R})$ symmetry and solvable multiboson systems

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INTRODUCTION

We extend the well known Holstein-Primakoff method to the case of $sl(2, \mathbb{R})$ algebra. The original method is based on investigation of quantum systems described by the elements of the enveloping algebra of Heisenberg-Weyl algebra satisfying commutation relations of $su(2)$, see [HP40], and is most effective in the study of shell model of the nucleus. Various examples of that can be found in [KM91]. Our generalization is used to the integration of the quantum-optical models such as for example the model of the two-level atom coupled to single mode radiation, see e.g. [SB84, Suk89].

In this paper we combine the $sl(2, \mathbb{R})$ symmetry and the theory of the orthogonal polynomials in order to integrate the wide family of multiboson Hamiltonians describing the interaction of bosons with non-linear medium as well as the boson-boson interaction. Hence among other one can use them for modelling the non-linear phenomena in quantum optics including Kerr-like effect and parametric generation, see [PL98, HCOT04].

The paper consists of three sections. In the Section 1 we distinguish the two-parameter family of one-mode multiboson Hamiltonians (1.38) defined as a linear combination of the generators of Lie algebra $sl(2, \mathbb{R})$ obtained by solving the difference equations (1.4)-(1.6). We define the group of the Bogoliubov-like transformations and using it we distinguish the classes of Hamiltonians related to the appropriate classes of orthogonal polynomials, see also Appendix A. At the end of the section, we construct the system of coherent states which allows us to find

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new, as we suppose, realization of the discrete series representation of the group $SL(2, \mathbb{R})$.

Section 2 is devoted to solving the class of two-mode Hamiltonians (2.9), which are defined by the combination of Casimirs for $sl(2, \mathbb{R})$ algebras. Similarly to the one-mode case the integration of (2.9) is carried out by the orthogonal polynomials, i.e by dual Hahn and continuous dual Hahn polynomials.

In the last section we discuss the possible physical interpretation of the quantum models governed by the Hamiltonians investigated in previous sections.

1. ONE-MODE SYSTEMS

1.1. Multiboson representations of $sl(2, \mathbb{R})$ in bosonic Fock space.

Let \mathbf{a}, \mathbf{a}^* be the standard annihilation and creation operators satisfying CCR relations and acting in the Hilbert space \mathcal{H} . Let $\{|n\rangle\}_{n=0}^{\infty}$ be the orthonormal basis in \mathcal{H} consisting of eigenvectors of number occupation operator $\mathbf{n} := \mathbf{a}^* \mathbf{a}$.

For any fixed $l \in \mathbb{N}$, we define the *multiboson representation* of $sl(2, \mathbb{R})$ as the triple of operators

$$\mathbf{A}_0 := \alpha_0(\mathbf{n}), \quad \mathbf{A}_- := \alpha_-(\mathbf{n}) \mathbf{a}^l, \quad \mathbf{A}_+ := (\mathbf{a}^*)^l \alpha_-(\mathbf{n}) \quad (1.1)$$

defined on dense subset \mathcal{D}_0 consisting of finite combinations of $|n\rangle$ and satisfying $sl(2, \mathbb{R})$ commutation relations:

$$[\mathbf{A}_-, \mathbf{A}_+] = \mathbf{A}_0, \quad [\mathbf{A}_0, \mathbf{A}_{\pm}] = \pm 2\mathbf{A}_{\pm} \quad (1.2)$$

with symmetricity conditions:

$$\mathbf{A}_0 \subset \mathbf{A}_0^*, \quad \mathbf{A}_- \subset \mathbf{A}_+^*, \quad \mathbf{A}_+ \subset \mathbf{A}_-^*. \quad (1.3)$$

We denote by \mathcal{A} the operator Lie algebra generated by $\mathbf{A}_0, \mathbf{A}_-, \mathbf{A}_+$.

Relations (1.2) and (1.3) imply that the functions α_0, α_- are real-valued and satisfy the following difference equations

$$(\alpha_0(n) - \alpha_0(n-l) - 2)\alpha_-(n-l) = 0 \quad \text{for } n \geq l, \quad (1.4)$$

$$(n+1)_l \alpha_-^2(n) - (n-l+1)_l \alpha_-^2(n-l) = \alpha_0(n) \quad \text{for } n \geq l, \quad (1.5)$$

$$(n+1)_l \alpha_-^2(n) = \alpha_0(n) \quad \text{for } 0 \leq n < l. \quad (1.6)$$

where $(n)_l = n(n+1)\dots(n+l-1)$ and one has applied the identities

$$(\mathbf{a}^*)^l \mathbf{a}^l = (\mathbf{n} - l + 1)_l \quad \mathbf{a}^l (\mathbf{a}^*)^l = (\mathbf{n} + 1)_l. \quad (1.7)$$

Without loss of generality we can assume that $\alpha_-(n) > 0$ for $n = 0, 1, \dots$ (by multiplying basis vectors by ± 1 and by removing common kernel of $\mathbf{A}_0, \mathbf{A}_-, \mathbf{A}_+$).

The solution to the system of difference equations (1.4), (1.5), (1.6) is of the form

$$\alpha_0(n) = 2 \left\lfloor \frac{n}{l} \right\rfloor + \alpha_0(n \bmod l), \quad (1.8a)$$

$$\alpha_-(n) = \sqrt{\frac{1}{(n+1)_l} \left(\left\lfloor \frac{n}{l} \right\rfloor + \alpha_0(n \bmod l) \right) \left(\left\lfloor \frac{n}{l} \right\rfloor + 1 \right)}, \quad (1.8b)$$

where $[x]$ is the integer part of x . The values $\alpha_0(r)$ for $r = 0, \dots, l-1$ are arbitrary positive constants corresponding to initial conditions on the solution α_0 of the difference equation (1.4).

In order to express the operators \mathbf{A}_0 , \mathbf{A}_- , \mathbf{A}_+ explicitly in terms of the creation and annihilation operators let us define the bounded operator

$$\mathbf{R} := \frac{l-1}{2} + \sum_{m=1}^{l-1} \frac{\exp(-\frac{2\pi im}{l} \mathbf{n})}{\exp(\frac{2\pi im}{l}) - 1} \quad (1.9)$$

for $l > 1$ and $\mathbf{R} := 0$ for $l = 1$, see [HCOT04]. This operator acts on elements of the basis by

$$\mathbf{R} |n\rangle = n \bmod l |n\rangle \quad (1.10)$$

and commutes with operators \mathbf{A}_0 , \mathbf{A}_- , \mathbf{A}_+ . Thus one has

$$\frac{1}{l} (\mathbf{n} - \mathbf{R}) |n\rangle = \left\lfloor \frac{n}{l} \right\rfloor |n\rangle. \quad (1.11)$$

Finally the multiboson representation of $sl(2, \mathbb{R})$ is given in terms of \mathbf{a} and \mathbf{a}^* by

$$\mathbf{A}_0 = \frac{2}{l} (\mathbf{n} - \mathbf{R}) + \alpha_0(\mathbf{R}), \quad (1.12a)$$

$$\mathbf{A}_- = \sqrt{\frac{1}{(\mathbf{n}+1)_l} \left(\frac{1}{l} (\mathbf{n} - \mathbf{R}) + \alpha_0(\mathbf{R}) \right) \left(\frac{1}{l} (\mathbf{n} - \mathbf{R}) + 1 \right)} \mathbf{a}^l. \quad (1.12b)$$

α_0 in these formulae is treated as an arbitrary positive function on spectrum of \mathbf{R} , i.e. it is defined by initial conditions $\alpha_0(r)$, $r \in \text{spec}(\mathbf{R}) = \{0, 1, \dots, l-1\}$, see (1.8).

The formulae (1.12) show that the Hilbert space \mathcal{H} splits

$$\mathcal{H} = \bigoplus_{r=0}^{l-1} \mathcal{H}_r \quad (1.13)$$

onto invariant subspaces

$$\mathcal{H}_r := \text{span}\{ |k\rangle_r := |kl+r\rangle \mid k \in \mathbb{N} \cup \{0\} \}, \quad (1.14)$$

which are eigenspaces of \mathbf{R} corresponding to the eigenvalue r .

Let us observe that $|k\rangle_r$ are eigenvectors of \mathbf{A}_0

$$\mathbf{A}_0 |k\rangle_r = (2k + \alpha_0(r)) |k\rangle_r \quad (1.15)$$

and \mathbf{A}_- , \mathbf{A}_+ act on $|k\rangle_r$ as weighted shift operators:

$$\mathbf{A}_- |k\rangle_r = \sqrt{k(k + \alpha_0(r) - 1)} |k - 1\rangle_r, \quad (1.16)$$

$$\mathbf{A}_+ |k\rangle_r = \sqrt{(k + \alpha_0(r))(k + 1)} |k + 1\rangle_r. \quad (1.17)$$

So, (1.13) gives the decomposition of the multiboson representation of $sl(2, \mathbb{R})$ onto irreducible components.

Let us now present two examples of the multiboson representation of $sl(2, \mathbb{R})$.

Example 1.1. Let us begin by the simplest case $l = 1$. Then $\mathbf{R} = 0$ and $\alpha_0(\mathbf{R}) = c$. Therefore

$$\mathbf{A}_0 = 2\mathbf{n} + c, \quad \mathbf{A}_- = \sqrt{\mathbf{n} + c} \mathbf{a}. \quad (1.18)$$

It is $sl(2, \mathbb{R})$ analogue of Holstein-Primakoff mapping for $SU(2)$, see [KM91].

□

Example 1.2. Let us put $l = 2$ and the special choice of α_0 , namely $\alpha_0(\mathbf{R}) = \frac{1}{2} + \mathbf{R}$. Then formulae (1.12) simplify to:

$$\mathbf{A}_0 = \mathbf{n} + \frac{1}{2}, \quad \mathbf{A}_- = \frac{1}{2} \mathbf{a}^2. \quad (1.19)$$

This representation is associated with the problem of second-harmonic generation, see [WT72, JD92].

□

The Casimir operator for \mathcal{A} is of the form

$$\mathbf{C}_{\mathcal{A}} := \frac{1}{2} \mathbf{A}_0^2 - (\mathbf{A}_- \mathbf{A}_+ + \mathbf{A}_+ \mathbf{A}_-) \quad (1.20)$$

and it can be explicitly expressed by

$$\mathbf{C}_{\mathcal{A}} = \frac{1}{2} \alpha_0(\mathbf{R}) (\alpha_0(\mathbf{R}) - 2). \quad (1.21)$$

Thus $\mathbf{C}_{\mathcal{A}}|_{\mathcal{H}_r} = \frac{1}{2} \alpha_0(r) (\alpha_0(r) - 2)$ and each \mathcal{H}_r is contained in some eigenspace of $\mathbf{C}_{\mathcal{A}}$. For $0 < \alpha_0(r) < 2$ or $\alpha_0(r) = 2, 3, \dots$ representation (1.15)-(1.17) corresponds respectively to the complementary and discrete series of unitary representations of the group $SL(2, \mathbb{R})$, see [VK91].

From (1.15) it follows that \mathbf{A}_0 is closable with closure defined on

$$\mathcal{D}_1 := \left\{ \sum_{n=0}^{\infty} v_n |n\rangle \in \mathcal{H} \mid \sum_{n=0}^{\infty} n^2 |v_n|^2 < \infty \right\}. \quad (1.22)$$

Equations (1.16)-(1.17) imply that \mathbf{A}_- and \mathbf{A}_+ are superpositions of diagonal operator and shift operator. Both of these diagonal operators are closable with closure defined on \mathcal{D}_1 . Since shift operator is bounded we conclude that the closures of all generators of \mathcal{A} are defined on common domain \mathcal{D}_1 . From now on \mathbf{A}_0 , \mathbf{A}_- and \mathbf{A}_+ will denote these closures. Moreover symmetricity conditions (1.3) are replaced by stronger conditions

$$\mathbf{A}_0 = \mathbf{A}_0^*, \quad \mathbf{A}_- = \mathbf{A}_+^*, \quad \mathbf{A}_+ = \mathbf{A}_-^*, \quad (1.23)$$

i.e. operator \mathbf{A}_0 is self-adjoint and \mathbf{A}_- , \mathbf{A}_+ are mutually adjoint.

1.2. Bogoliubov-like transformations.

Let us consider a linear transformation of \mathcal{A} preserving commutation relations (1.2) and conditions (1.23). We will call such transformation a *Bogoliubov transformation* for the algebra $sl(2, \mathbb{R})$ in the analogy to Bogoliubov transformations for Heisenberg-Weyl algebra, see [Bog82].

The group of all transformations of this type is isomorphic to the group $\mathfrak{B} := \mathbb{R}^\times \rtimes \mathbb{Z}_2$, where $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{Z}_2 = \{-1, 1\}$, with the group operation defined by

$$(a, \sigma) \cdot (b, \tau) := (ab^\sigma, \sigma\tau). \quad (1.24)$$

One shows that the transformation given by an element $(a, \sigma) \in \mathfrak{B}$ acts on the generators of \mathcal{A} in the following way

$$\begin{aligned} \mathfrak{b}_{a,\sigma}(\mathbf{A}_0) &:= \frac{1+a^2}{2a} \mathbf{A}_0 + \sigma \frac{1-a^2}{2a} (\mathbf{A}_- + \mathbf{A}_+), \\ \mathfrak{b}_{a,\sigma}(\mathbf{A}_-) &:= \frac{1-a^2}{4a} \mathbf{A}_0 + \sigma \frac{(1-a)^2}{4a} \mathbf{A}_+ + \sigma \frac{(1+a)^2}{4a} \mathbf{A}_-, \\ \mathfrak{b}_{a,\sigma}(\mathbf{A}_+) &:= \frac{1-a^2}{4a} \mathbf{A}_0 + \sigma \frac{(1+a)^2}{4a} \mathbf{A}_+ + \sigma \frac{(1-a)^2}{4a} \mathbf{A}_-. \end{aligned} \quad (1.25)$$

Let us notice that the domain of operators (1.25) is also \mathcal{D}_1 and the Casimir operator (1.20) is invariant with respect to the action (1.25).

There exists the unitary projective representation $(a, \sigma) \mapsto \mathbb{U}_{a,\sigma}$ in \mathcal{H}_r of the subgroup $\mathbb{R}_+ \rtimes \mathbb{Z}_2 \subset \mathfrak{B}$ which implements the action of $\mathfrak{b}_{a,\sigma}$, i.e.

$$\mathfrak{b}_{a,\sigma}(\mathbf{X}) = \mathbb{U}_{a,\sigma} \mathbf{X} \mathbb{U}_{a,\sigma}^* \quad (1.26)$$

for all $\mathbf{X} \in \mathcal{A}$ on \mathcal{D}_1 .

In order to prove this we give the explicit formula for $\mathbb{U}_{a,\sigma}$. First of all let us observe that basis vectors $|k\rangle_r$ in \mathcal{H}_r are eigenvectors of \mathbf{A}_0 . We will show that $\mathbf{b}_{a,\sigma}(\mathbf{A}_0)$ has the same spectrum as \mathbf{A}_0 and we will compute its eigenvectors. To this end we observe that

$$\mathbf{b}_{a,\sigma}(\mathbf{A}_0) |k\rangle_r = b_{k-1} |k-1\rangle_r + a_k |k\rangle_r + b_k |k+1\rangle_r, \quad (1.27)$$

where

$$a_k = \frac{a^{-\sigma} + a^\sigma}{2} (2k + \alpha_0(r)) \quad (1.28a)$$

$$b_k = \frac{a^{-\sigma} - a^\sigma}{2} \sqrt{(k + \alpha_0(r))(k + 1)}. \quad (1.28b)$$

If $a \neq 1$ then the formula (1.27) is directly related to three term recurrence relation

$$xP_k(x) = b_{k-1} P_{k-1}(x) + a_k P_k(x) + b_k P_{k+1}(x) \quad (1.29)$$

which is valid for any orthonormal polynomials family $\{P_n\}_{n=0}^\infty$ for appropriate choice of coefficients.

In case when coefficients a_k and b_k are given by (1.28) we obtain Meixner polynomials $P_n(x) = M_k(x; \alpha_0(r), c)$, where

$$c = \left(\frac{a-1}{a+1} \right)^2, \quad (1.30)$$

see [Akh65, KS98]. From this it follows that

$$|n; a, \sigma\rangle_r := \begin{cases} \sigma^n \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} M_k(n; \alpha_0(r), c) |k\rangle_r & \text{for } a^\sigma < 1 \\ \sigma^n \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} (-1)^k M_k(n; \alpha_0(r), c) |k\rangle_r & \text{for } a^\sigma > 1 \\ \sigma^n |n\rangle_r & \text{for } a = 1 \end{cases} \quad (1.31)$$

and the corresponding eigenvalues are

$$E_n^{(\sigma, a)} := 2n + \alpha_0(r). \quad (1.32)$$

Since the spectra of \mathbf{A}_0 and $\mathbf{b}_{a,\sigma}(\mathbf{A}_0)$ coincide we define the unitary operator $\mathbb{U}_{a,\sigma}$ by

$$\mathbb{U}_{a,\sigma} |n\rangle_r := |n; a, \sigma\rangle_r. \quad (1.33)$$

Using the fact that $\mathbf{b}_{a,\sigma}(\mathbf{A}_-)$ and $\mathbf{b}_{a,\sigma}(\mathbf{A}_+)$ satisfy the commutation relations (1.2), we can check that they act on eigenvectors $|n; a, \sigma\rangle_r$ as weighted shift operators:

$$\mathbf{b}_{a,\sigma}(\mathbf{A}_-) |n; a, \sigma\rangle_r = \sqrt{n(n + \alpha_0(r) - 1)} |n-1; a, \sigma\rangle_r, \quad (1.34)$$

$$\mathfrak{b}_{a,\sigma}(\mathbf{A}_+) |n; a, \sigma\rangle_r = \sqrt{(n + \alpha_0(r))(n + 1)} |n + 1; a, \sigma\rangle_r. \quad (1.35)$$

Thus the unitary operator $\mathbb{U}_{a,\sigma}$ defined by (1.33) satisfies (1.26) on dense subset of \mathcal{H}_r consisting of finite linear combinations of vectors $|n; b, \pi\rangle_r$ for $n \in \mathbb{N} \cup \{0\}$ and $(b, \pi) \in \mathbb{R}_+ \rtimes \mathbb{Z}_2$. In such way we have constructed the unitary representation of the group $\mathbb{R}_+ \rtimes \mathbb{Z}_2$, which implements the action (1.25). However, it is not possible to extend it to whole \mathfrak{B} since $\mathfrak{b}_{-a,\sigma}$ can be decomposed into

$$\mathfrak{b}_{-a,\sigma} = \mathfrak{b}_{a,\sigma} \circ \mathfrak{b}_{-1,1}, \quad (1.36)$$

where

$$\mathfrak{b}_{-1,1}(\mathbf{X}) = -\mathbf{X} \quad (1.37)$$

for $\mathbf{X} \in \mathcal{A}$.

1.3. Integrable one-mode Hamiltonians.

The aim of this section is to integrate quantum system described by arbitrary self-adjoint operator belonging to the multiboson algebra \mathcal{A} :

$$\mathbf{H}_{\mu\nu} := \frac{\mu + \nu}{2} \mathbf{A}_0 + \frac{\mu - \nu}{2} (\mathbf{A}_- + \mathbf{A}_+), \quad (1.38)$$

where $(\mu, \nu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Let us observe that

$$\mathbf{H}_{\mu\nu} |k\rangle_r = b_{k-1} |k-1\rangle_r + a_k |k\rangle_r + b_k |k+1\rangle_r, \quad (1.39)$$

where

$$a_k = \frac{\mu + \nu}{2} (2k + \alpha_0(r)) \quad (1.40a)$$

$$b_k = \frac{\mu - \nu}{2} \sqrt{(k + \alpha_0(r))(k + 1)}. \quad (1.40b)$$

If $\mu \neq \nu$ the formula (1.39) is directly related to three term recurrence relation (1.29) for some family of orthonormal polynomials $\{P_n\}_{n=0}^\infty$. Since $\sum \frac{1}{b_k}$ is divergent then there exists the unique measure $d\omega$ on \mathbb{R} such the map F given by

$$\mathcal{H}_r \ni |k\rangle_r \longmapsto F(|k\rangle_r) := P_k \in L^2(\mathbb{R}, d\omega) \quad (1.41)$$

is the isomorphism of Hilbert spaces with the property that

$$F \circ \mathbf{H}_{\mu\nu}|_{\mathcal{H}_r} \circ F^{-1} = \hat{x}, \quad (1.42)$$

where \hat{x} is the operator of multiplication in $L^2(\mathbb{R}, d\omega)$, see [Chi78, OHT01]. Thus we gather that the spectrum $\text{spec}(\mathbf{H}_{\mu\nu})$ is the support of measure $d\omega$. It means that by finding the measure $d\omega$ and

polynomials P_n we obtain the evolution flow

$$R \ni t \longrightarrow e^{it\mathbf{H}_{\mu\nu}} = F^{-1} \circ e^{it\hat{x}} \circ F \in \text{Aut}(\mathcal{H}_r) \quad (1.43)$$

of quantum system described by the Hamiltonian $\mathbf{H}_{\mu\nu}$.

From definition of Bogoliubov group \mathfrak{B} it follows that transformations $\mathfrak{b}_{a,\sigma}$ preserve the family of operators $\mathbf{H}_{\mu\nu}$ and the labels (μ, ν) transform as follows

$$(a, 1) : (\mu, \nu) \mapsto (a^{-1}\mu, a\nu), \quad (a, -1) : (\mu, \nu) \mapsto (a\nu, a^{-1}\mu). \quad (1.44)$$

This defines the action of the group \mathfrak{B} in the set $\mathbb{R}^2 \setminus \{(0, 0)\}$ of labels. We conclude that orbits of \mathfrak{B} are pairs of hiperbolae indexed by one real parameter $c \in \mathbb{R}$

$$\mathcal{O}_c := \mathfrak{B} \cdot (c, 1) = \{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \mid xy = c\}. \quad (1.45)$$

We can restrict our considerations to each component \mathcal{H}_r of decomposition (1.13) separately since they are invariant under the action of $\mathbf{H}_{\mu\nu}$. Due to the implementation formula (1.26) it is sufficient to find spectral decomposition for one operator from each orbit (1.45), e.g. $\mathbf{H}_{\sqrt{c}, \sqrt{c}}$, $\mathbf{H}_{\sqrt{c}, -\sqrt{c}}$ and $\mathbf{H}_{1,0}$. Taking into account scaling by constant we can further restrict ourselves to three Hamiltonians $\mathbf{H}_{1,1}$, $\mathbf{H}_{1,-1}$ and $\mathbf{H}_{1,0}$.

Since $\mathbf{H}_{1,1} = \mathbf{A}_0$ then the spectral problem is trivial and is solved by formula (1.15).

For $\mathbf{H}_{1,0}$ formula (1.29) defines Laguerre orthonormal polynomials

$$P_n(x) = L_n^{(\alpha_0(r)-1)}(2x) \quad (1.46)$$

and measure $d\omega$ is given by

$$d\omega(x) = 2(2x)^{\alpha_0(r)-1} e^{-2x} \theta(x) dx, \quad (1.47)$$

where θ is Heaviside function. Thus the spectrum $\text{spec}(\mathbf{H}_{1,0}) = \mathbb{R}_+ \cup \{0\}$.

For $\mathbf{H}_{1,-1}$ formula (1.29) defines Meixner-Pollaczek orthonormal polynomials

$$P_n(x) = P_n^{(\frac{\alpha_0(r)}{2})} \left(\frac{x}{2}; \frac{\pi}{2} \right) \quad (1.48)$$

and measure is $d\omega$ is given by

$$d\omega(x) = \frac{1}{2} \left| \Gamma \left(\frac{1}{2} \alpha_0(r) + i \frac{x}{2} \right) \right|^2 dx \quad (1.49)$$

and the spectrum $\text{spec}(\mathbf{H}_{1,-1}) = \mathbb{R}$, see [KS98].

In Appendix A we present complete list (for all μ, ν) of Hilbert spaces $L^2(\mathbb{R}, d\omega)$ in which $\mathbf{H}_{\mu\nu}$ act as \hat{x} . In Figure 1 we have illustrated the decomposition of (μ, ν) -plane into sectors corresponding to different

families of orthogonal polynomials. Let us remark that in the first quadrant the spectra of $\mathbf{H}_{\mu\nu}$ are discrete and bounded from below. On the boundary of this quadrant the spectra are $\mathbb{R}_+ \cup \{0\}$. In the third quadrant all spectra are also discrete but bounded from above. On the boundary of this quadrant the spectra are $\mathbb{R}_- \cup \{0\}$. In the second and fourth quadrants the spectra are \mathbb{R} .

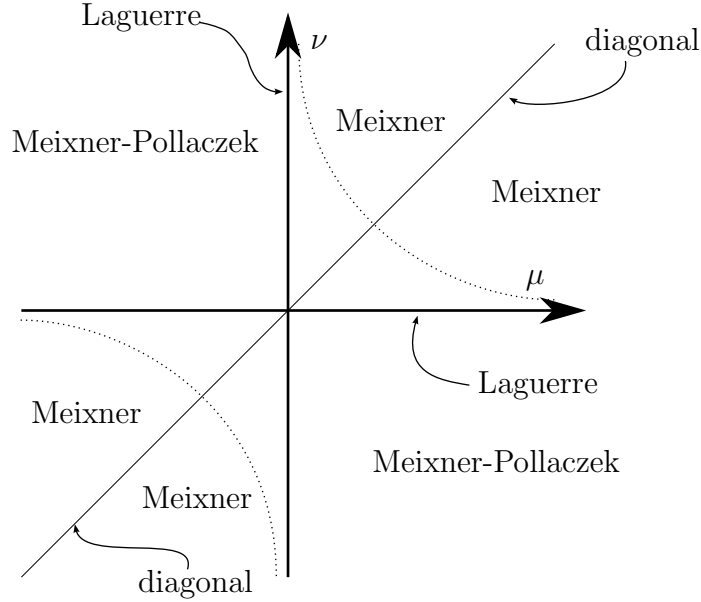


FIGURE 1. Orthogonal polynomials assigned to $\mathcal{H}_{\mu\nu}$

1.4. Coherent state representation.

The goal of this section is to give the coherent state representation of the Hamiltonians $\mathbf{H}_{\mu\nu}$ and flows $e^{it\mathbf{H}_{\mu\nu}}$ generated by them.

We will consider coherent states as eigenstates of \mathbf{A}_- , see [Odz98]. Due to the decomposition (1.13) into irreducible representations, it is sufficient to restrict our considerations to each \mathcal{H}_r separately

$$\mathbf{A}_- |\zeta\rangle_r = \zeta |\zeta\rangle_r. \quad (1.50)$$

From (1.16) we see that the coherent states $|\zeta\rangle_r \in \mathcal{H}_r$ are given by the series

$$|\zeta\rangle_r := \sum_{k=0}^{\infty} \frac{\zeta^k}{\sqrt{k!(\alpha_0(r))_k}} |k\rangle_r \quad (1.51)$$

which converges for any $\zeta \in \mathbb{C}$ and belongs to domain \mathcal{D}_1 .

The notion of the coherent states allows us to construct the anti-unitary embedding

$$\mathcal{H}_r \ni |\psi\rangle \longmapsto I_r(\psi)(\zeta) := \langle \psi | \zeta \rangle_r \in L^2\mathcal{O}(\mathbb{C}, d\mu_r) \quad (1.52)$$

of \mathcal{H}_r into the Hilbert space $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ of holomorphic functions on \mathbb{C} , which are square integrable with respect to the measure

$$d\mu_r(\zeta, \bar{\zeta}) := \frac{\rho^{\alpha_0(r)} K_{\alpha_0(r)}(2\rho)}{2\pi\Gamma(\alpha_0(r))} \rho \, d\rho \, d\phi, \quad (1.53)$$

where $\zeta = \rho e^{i\phi}$ and $K_{\alpha_0(r)}$ is the modified Bessel function of the second kind, see [OS97]. The space $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ has the reproducing kernel

$$\mathcal{K}(\bar{\eta}, \zeta) := \langle \eta | \zeta \rangle = {}_0F_1 \left(\begin{matrix} - \\ \alpha_0(r) \end{matrix} \middle| \bar{\eta}\zeta \right), \quad (1.54)$$

i.e. for any $f \in L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ one has

$$\int_{\mathbb{C}} \mathcal{K}(\bar{\eta}, \zeta) f(\eta) d\mu_r(\eta, \bar{\eta}) = f(\zeta). \quad (1.55)$$

The isomorphism I_r gives the realization of the operators \mathbf{A}_0 , \mathbf{A}_+ , \mathbf{A}_- as the differential operators

$$I_r \circ \mathbf{A}_0 \circ I_r^{-1} = 2\zeta \frac{d}{d\zeta} + \alpha_0(r), \quad (1.56)$$

$$I_r \circ \mathbf{A}_+ \circ I_r^{-1} = \zeta, \quad (1.57)$$

$$I_r \circ \mathbf{A}_- \circ I_r^{-1} = \left(\alpha_0(r) + \zeta \frac{d}{d\zeta} \right) \frac{d}{d\zeta} \quad (1.58)$$

acting in $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$. In order to describe them as the generators of the discrete series $\alpha_0(r) = 2, 3, \dots$ representation of the group $SL(2, \mathbb{R})$, let us consider a unitary integral transform

$$\mathcal{P} : L^2\mathcal{O}(\mathbb{C}, d\mu_r) \longrightarrow L^2\mathcal{O}(\mathbb{D}, d\nu_r), \quad (1.59)$$

where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and

$$d\nu_r(z, \bar{z}) := \frac{\alpha_0(r) - 1}{\pi} (1 - |z|^2)^{\alpha_0(r)-2} d^2z, \quad (1.60)$$

given by

$$\mathcal{P}f(z) := \int_{\mathbb{C}} e^{z\bar{\zeta}} f(\zeta) d\mu_r(\zeta, \bar{\zeta}). \quad (1.61)$$

Using (1.61) we find that in the space $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$ the operators (1.56)-(1.58) are given by

$$\begin{aligned}\mathcal{P} \circ I_r \circ \mathbf{A}_0 \circ I_r^{-1} \circ \mathcal{P}^{-1} &= 2z \frac{d}{dz} + \alpha_0(r) \\ \mathcal{P} \circ I_r \circ \mathbf{A}_+ \circ I_r^{-1} \circ \mathcal{P}^{-1} &= z^2 \frac{d}{dz} + \alpha_0(r)z \\ \mathcal{P} \circ I_r \circ \mathbf{A}_- \circ I_r^{-1} \circ \mathcal{P}^{-1} &= \frac{d}{dz}\end{aligned}\quad (1.62)$$

and they are the generators of the discrete series representation

$$U_g^{\alpha_0(r)} \phi(z) = (bz + \bar{a})^{-\alpha_0(r)} \phi\left(\frac{az + \bar{b}}{bz + \bar{a}}\right) \quad (1.63)$$

of the group $SL(2, \mathbb{R})$ in $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$, see [Per86]. Here we have identified $SL(2, \mathbb{R})$ with $SU(1, 1)$ using the isomorphism

$$SL(2, \mathbb{R}) \ni g \longleftrightarrow \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} := \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} g \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SU(1, 1). \quad (1.64)$$

From (1.63) we obtain the representation of $SL(2, \mathbb{R})$ in $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ in the integral form

$$T_g^{\alpha_0(r)} f(\zeta) := \int_{\mathbb{C}} \mathfrak{K}_g^{\alpha_0(r)}(\zeta, \bar{\eta}) f(\eta) d\mu_r(\eta, \bar{\eta}) \quad (1.65)$$

with the kernel

$$\mathfrak{K}_g^{\alpha_0(r)}(\zeta, \bar{\eta}) := \int_{\mathbb{D}} e^{w\bar{\eta} - \zeta \frac{a\bar{w} - b}{b\bar{w} - \bar{a}}} (-\bar{b}\bar{w} + \bar{a})^{\alpha_0(r)} d\nu_r(w, \bar{w}). \quad (1.66)$$

In order to obtain the explicit formula for the evolution operators $e^{i\mathbf{H}_{\mu\nu}t}$ let us consider the corresponding family of one-parameter subgroups of $SL(2, \mathbb{R})$

$$g_{\mu\nu}(t) := \cos(\sqrt{\mu\nu} t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(\sqrt{\mu\nu} t)}{2\sqrt{\mu\nu}} \begin{pmatrix} -\mu + \nu & \mu + \nu \\ -\mu - \nu & \mu - \nu \end{pmatrix} \quad (1.67)$$

The infinitesimal generator of the action $T_{g_{\mu\nu}(t)}^{\alpha_0(r)}$ on $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ is $I_r \circ (i\mathbf{H}_{\mu\nu}t) \circ I_r^{-1}$. Thus the time evolution operator $I_r \circ (e^{i\mathbf{H}_{\mu\nu}t}) \circ I_r^{-1}$ acts in $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ as the integral operator (1.65) with the kernel

$$\mathfrak{K}_{g_{\mu\nu}(t)}^{\alpha_0(r)}(\zeta, \bar{\eta}) := \int_{\mathbb{D}} e^{w\bar{\eta} - \zeta \frac{a_{\mu\nu}(t)\bar{w} - b_{\mu\nu}(t)}{b_{\mu\nu}(t)\bar{w} - a_{\mu\nu}(t)}} \left(-\overline{b_{\mu\nu}(t)}\bar{w} + \overline{a_{\mu\nu}(t)} \right)^{\alpha_0(r)} d\nu_r(w, \bar{w}), \quad (1.68)$$

where

$$a_{\mu\nu}(t) = \cos(\sqrt{\mu\nu} t) + \frac{i \sin(\sqrt{\mu\nu} t) (\mu + \nu)}{2\sqrt{\mu\nu}} \quad (1.69)$$

$$b_{\mu\nu}(t) = \frac{i \sin(\sqrt{\mu\nu} t) (-\mu + \nu)}{2\sqrt{\mu\nu}}. \quad (1.70)$$

Ending let us remark that in $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ the Hamiltonian $\mathbf{H}_{\mu\nu}$ is represented as a second order differential operator

$$I_r \circ \mathbf{H}_{\mu\nu} \circ I_r^{-1} = \frac{\mu + \nu}{2} \left(2\zeta \frac{d}{d\zeta} + \alpha_0(r) \right) + \frac{\mu - \nu}{2} \left(\zeta + \left(\alpha_0(r) + \zeta \frac{d}{d\zeta} \right) \frac{d}{d\zeta} \right) \quad (1.71)$$

and in $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$ as a first order differential operator

$$\mathcal{P} \circ I_r \circ \mathbf{H}_{\mu\nu} \circ I_r^{-1} \circ \mathcal{P}^{-1} = \frac{\mu + \nu}{2} \left(2z \frac{d}{dz} + \alpha_0(r) \right) + \frac{\mu - \nu}{2} \left((z^2 + 1) \frac{d}{dz} + \alpha_0(r) z \right). \quad (1.72)$$

In Section 1.3 it was shown that this operator has discrete spectrum for $\mu\nu > 0$ The eigenproblem

$$I_r \circ \mathbf{H}_{\mu\nu} \circ I_r^{-1} \psi = \lambda \psi \quad (1.73)$$

in the Hilbert space $L^2\mathcal{O}(\mathbb{C}, d\mu_r)$ leads to the degenerate Gauss equation and its solutions can be expressed in terms of confluent hypergeometric function and Laguerre-type function, see [Smi61]. On the other hand the eigenproblem

$$\mathcal{P} \circ I_r \circ \mathbf{H}_{\mu\nu} \circ I_r^{-1} \circ \mathcal{P}^{-1} \phi = \lambda \phi \quad (1.74)$$

in the Hilbert space $L^2\mathcal{O}(\mathbb{D}, d\nu_r)$ leads to the first order differential equation and its solution in the case $\mu > \nu > 0$ for the eigenvalue λ is given by

$$\phi(z) = C \left(z - + \frac{(\sqrt{\mu} - \sqrt{\nu})^2}{\mu - \nu} \right)^A \left(z + \frac{(\sqrt{\mu} + \sqrt{\nu})^2}{\mu - \nu} \right)^B, \quad (1.75)$$

where

$$A = \frac{\lambda(\mu - \nu)}{2\sqrt{\mu\nu}} - \frac{\alpha_0(r)}{2} \quad B = \frac{-\lambda(\mu - \nu)}{2\sqrt{\mu\nu}} - \frac{\alpha_0(r)}{2}. \quad (1.76)$$

Thus

$$\psi = (\mathcal{P} \circ I_r)^{-1} \phi \quad (1.77)$$

and this kind of relationship gives the various integral representations of corresponding special functions. However the investigation of these problems is not the goal of the paper.

2. TWO-MODE SYSTEMS

2.1. Direct sum of multiboson $sl(2, \mathbb{R})$ algebras.

In this section we will work within two-mode bosonic Fock space $\mathcal{H} \otimes \mathcal{H}$ with the standard annihilation, creation operators $\mathbf{a}_i, \mathbf{a}_i^*, i = 0, 1$, acting in i^{th} mode and with the orthonormal basis $\{|n_0, n_1\rangle\}_{n_0, n_1=0}^\infty$ consisting of eigenvectors of $\mathbf{n}_i = \mathbf{a}_i^* \mathbf{a}_i$.

Let us consider the two-mode analogue of the investigation presented in the previous section. Let $\mathcal{A} \oplus \mathcal{B} \cong so(2, 2)$ be the direct sum of two multiboson $sl(2, \mathbb{R})$ operator algebras $\mathcal{A} := \text{span}\{\mathbf{A}_0, \mathbf{A}_-, \mathbf{A}_+\}$ and $\mathcal{B} := \text{span}\{\mathbf{B}_0, \mathbf{B}_-, \mathbf{B}_+\}$ acting on $\mathcal{D}_0 \otimes \mathcal{D}_0 \subset \mathcal{H} \otimes \mathcal{H}$, where

$$\mathbf{A}_0 := \alpha_0(\mathbf{n}_0, \mathbf{n}_1) \quad \mathbf{A}_- := \alpha_-(\mathbf{n}_0, \mathbf{n}_1) \mathbf{a}_0^{l_0} \quad \mathbf{A}_+ := (\mathbf{a}_0^*)^{l_0} \alpha_-(\mathbf{n}_0, \mathbf{n}_1) \quad (2.1)$$

$$\mathbf{B}_0 := \beta_0(\mathbf{n}_0, \mathbf{n}_1) \quad \mathbf{B}_- := \beta_-(\mathbf{n}_0, \mathbf{n}_1) \mathbf{a}_1^{l_1} \quad \mathbf{B}_+ := (\mathbf{a}_1^*)^{l_1} \beta_-(\mathbf{n}_0, \mathbf{n}_1) \quad (2.2)$$

and \mathcal{A} and \mathcal{B} satisfy (1.2) and (1.3)

Proceeding as in previous section we gather that the commutation relations for $\mathcal{A} \oplus \mathcal{B}$ imply the difference equations on functions $\alpha_0, \alpha_-, \beta_0, \beta_-$. The solution of these equation gives the following expressions on operators:

$$\mathbf{A}_0 = \frac{2}{l_0} (\mathbf{n}_0 - \mathbf{R}_0) + \alpha_0(\mathbf{R}_0, \mathbf{R}_1) \quad (2.3a)$$

$$\mathbf{A}_- = \sqrt{\frac{1}{(\mathbf{n}_0 + 1)_{l_0}} \left(\frac{1}{l_0} (\mathbf{n}_0 - \mathbf{R}_0) + \alpha_0(\mathbf{R}_0, \mathbf{R}_1) \right) \left(\frac{1}{l_0} (\mathbf{n}_0 - \mathbf{R}_0) + 1 \right)} \mathbf{a}_0^{l_0}, \quad (2.3b)$$

$$\mathbf{B}_0 = \frac{2}{l_1} (\mathbf{n}_1 - \mathbf{R}_1) + \beta_0(\mathbf{R}_0, \mathbf{R}_1) \quad (2.4a)$$

$$\mathbf{B}_- = \sqrt{\frac{1}{(\mathbf{n}_1 + 1)_{l_1}} \left(\frac{1}{l_1} (\mathbf{n}_1 - \mathbf{R}_1) + \beta_0(\mathbf{R}_0, \mathbf{R}_1) \right) \left(\frac{1}{l_1} (\mathbf{n}_1 - \mathbf{R}_1) + 1 \right)} \mathbf{a}_1^{l_1}, \quad (2.4b)$$

where $\mathbf{R}_i, i = 0, 1$, are defined by (1.9) for i^{th} mode and α_0, β_0 are arbitrary positive functions on the product of spectra $(\text{spec } \mathbf{R}_0) \times (\text{spec } \mathbf{R}_1)$.

We observe that \mathbf{R}_i commute with $\mathcal{A} \oplus \mathcal{B}$ and that Casimir operators $\mathbf{C}_\mathcal{A}, \mathbf{C}_\mathcal{B}$ (see (1.20)) can be expressed as functions (not invertible in

general) of \mathbf{R}_0 and \mathbf{R}_1 . Thus we will decompose two-mode Fock space

$$\mathcal{H} \otimes \mathcal{H} = \bigoplus_{r_0=0}^{l_0-1} \bigoplus_{r_1=0}^{l_1-1} \mathcal{H}_{r_0, r_1}, \quad (2.5)$$

into common eigenspaces of \mathbf{R}_i

$$\mathcal{H}_{r_0, r_1} := \text{span}\{ |k_0, k_1\rangle_{r_0, r_1} := |k_0 l_0 + r_0, k_1 l_1 + r_1\rangle \mid k_0, k_1 \in \mathbb{N} \cup \{0\} \}, \quad (2.6)$$

on which $\mathcal{A} \oplus \mathcal{B}|_{\mathcal{H}_{r_0, r_1}}$ acts irreducibly.

Comparing the formulae (2.3)-(2.4) to (1.12) we extend all generators of $\mathcal{A} \oplus \mathcal{B}$ to the common domain $\mathcal{D}_1 \otimes \mathcal{D}_1$.

2.2. Hamiltonians.

Let us consider a class of $sl(2, \mathbb{R})$ subalgebras $\mathcal{D} := \text{span}\{\mathbf{D}_0, \mathbf{D}_-, \mathbf{D}_+\} \subset \mathcal{A} \oplus \mathcal{B}$ defined by

$$\begin{aligned} \mathbf{D}_0 &:= \mathfrak{b}_{a, \sigma}(\mathbf{A}_0) + \mathfrak{b}_{b, \tau}(\mathbf{B}_0), \\ \mathbf{D}_- &:= \mathfrak{b}_{a, \sigma}(\mathbf{A}_-) + \mathfrak{b}_{b, \tau}(\mathbf{B}_-), \\ \mathbf{D}_+ &:= \mathfrak{b}_{a, \sigma}(\mathbf{A}_+) + \mathfrak{b}_{b, \tau}(\mathbf{B}_+), \end{aligned} \quad (2.7)$$

for $(a, \sigma), (b, \tau) \in \mathfrak{B}$. For simplicity we have omitted indices in $\mathbf{D}_0, \mathbf{D}_-, \mathbf{D}_+$ corresponding to the elements of \mathfrak{B} .

Since $\mathbf{D}_0, \mathbf{D}_-, \mathbf{D}_+$ satisfy (1.2), the Casimir for this subalgebra is of the form

$$\mathbf{C}_{\mathcal{D}} = \frac{1}{2} \mathbf{D}_0^2 - \mathbf{D}_- \mathbf{D}_+ - \mathbf{D}_+ \mathbf{D}_- \quad (2.8)$$

and it commutes with the Casimirs $\mathbf{C}_{\mathcal{A}}$ and $\mathbf{C}_{\mathcal{B}}$.

Main goal of this section is to integrate the two-mode bosonic quantum system described by the Hamiltonian

$$\mathbf{H} := \frac{1}{2} (\mathbf{C}_{\mathcal{D}} - \mathbf{C}_{\mathcal{A}} - \mathbf{C}_{\mathcal{B}}). \quad (2.9)$$

Casimir operator $\mathbf{C}_{\mathcal{D}}$ is not defined on $\mathcal{D}_1 \otimes \mathcal{D}_1$ but due to the explicit formula

$$\begin{aligned} \mathbf{H} &= \frac{(a^2 + b^2)}{4ab} \mathbf{A}_0 \mathbf{B}_0 - \\ &- \sigma \tau \frac{(a - b)^2}{4ab} (\mathbf{A}_+ \mathbf{B}_+ + \mathbf{A}_- \mathbf{B}_-) - \sigma \frac{a^2 - b^2}{4ab} (\mathbf{A}_+ \mathbf{B}_0 + \mathbf{A}_- \mathbf{B}_0) + \\ &+ \tau \frac{a^2 - b^2}{4ab} (\mathbf{A}_0 \mathbf{B}_- + \mathbf{A}_0 \mathbf{B}_+) - \sigma \tau \frac{(a + b)^2}{4ab} (\mathbf{A}_+ \mathbf{B}_- + \mathbf{A}_- \mathbf{B}_+) \end{aligned} \quad (2.10)$$

we see that the symmetric operator \mathbf{H} is well defined on $\mathcal{D}_1 \otimes \mathcal{D}_1$.

Besides obvious integral of motions $\mathbf{R}_0, \mathbf{R}_1$ this Hamiltonian also commutes with \mathbf{D}_0 which is so called Manley-Rowe integral of motion

for \mathbf{H} . It means that the investigation of the spectral decomposition of \mathbf{H} can be reduced to the eigenspaces of \mathbf{D}_0 in each \mathcal{H}_{r_0, r_1} .

Due to the implementation formula (1.26) we can unitarily transform \mathbf{H} by $\mathbb{U}_{|a|^\sigma, -\sigma} \otimes \text{id} + \text{id} \otimes \mathbb{U}_{|b|^{-\tau}, \tau}$ to the Hamiltonian

$$\mathbf{H}_D := \frac{1}{2} \mathbf{A}_0 \mathbf{B}_0 + \mathbf{A}_+ \mathbf{B}_+ + \mathbf{A}_- \mathbf{B}_- \quad (2.11)$$

for $ab > 0$ and to the Hamiltonian

$$\mathbf{H}_C := \frac{1}{2} \mathbf{A}_0 \mathbf{B}_0 + \mathbf{A}_- \mathbf{B}_+ + \mathbf{A}_+ \mathbf{B}_- \quad (2.12)$$

for $ab < 0$ which correspond to indices $(a, \sigma), (b, \tau)$ in (2.7) being $(1, -1), (1, 1)$ or $(-1, -1), (-1, 1)$ respectively.

Spectral decompositions of self-adjoint extensions of (2.11) and (2.12) are essentially different thus we will discuss it separately. In order to simplify the notation, we will replace symbols $\alpha_0(r_0, r_1), \beta_0(r_0, r_1)$ by α_0 and β_0 in all formulae below when there will be no risk of confusion.

2.3. Spectral decomposition of \mathbf{H}_D .

In this case the operator \mathbf{D}_0 assumes the form

$$\mathbf{D}_0 = \mathbf{A}_0 + \mathbf{B}_0. \quad (2.13)$$

Thus its spectrum consists of numbers $2K + \alpha_0 + \beta_0$ for $K = 0, 1, \dots$ and corresponding eigenspaces

$$\mathcal{H}_{r_0, r_1}^K := \text{span}\{ |k\rangle_{r_0, r_1}^K \mid k = 0, 1, \dots, K \} \quad (2.14)$$

are $(K + 1)$ -dimensional, where

$$|k\rangle_{r_0, r_1}^K := |k, K - k\rangle_{r_0, r_1}. \quad (2.15)$$

So we get the decomposition

$$\mathcal{H}_{r_0, r_1} = \bigoplus_{K=0}^{\infty} \mathcal{H}_{r_0, r_1}^K. \quad (2.16)$$

onto subspaces invariant with respect to \mathbf{H}_D . Hamiltonian $\mathbf{H}_D|_{\mathcal{H}_{r_0, r_1}^K}$ is self-adjoint and thus \mathbf{H}_D admits unique self-adjoint extension in $\mathcal{H} \otimes \mathcal{H}$. Moreover it acts in basis $\{|k\rangle_{r_0, r_1}^K\}_{k=0}^K$ as follows

$$\mathbf{H}_D |k\rangle_{r_0, r_1}^K = b_{k-1} |k-1\rangle_{r_0, r_1}^K + a_k |k\rangle_{r_0, r_1}^K + b_k |k+1\rangle_{r_0, r_1}^K, \quad (2.17)$$

where

$$a_k = \frac{1}{2}(2k + \alpha_0)(2(K - k) + \beta_0), \quad (2.18)$$

$$b_k = \sqrt{(k+1)(k + \alpha_0)(K - k)(K - k + \beta_0)}. \quad (2.19)$$

Thus, analogously to the Section 1.3, the problem of finding the spectral decomposition of \mathbf{H}_D can be solved by means of some family of orthogonal polynomials. In this case the eigenvalues of $\mathbf{H}_D|_{\mathcal{H}_{r_0, r_1}^K}$ are given by

$$E_n^D := n(n + \alpha_0 + \beta_0 - 1) + \frac{1}{2}\alpha_0\beta_0 \quad (2.20)$$

for $n = 0, \dots, K$ and corresponding eigenvectors assume the form

$$|E_n^D\rangle_{r_0, r_1}^K = \sum_{k=0}^K P_k(n) |k\rangle_{r_0, r_1}^K, \quad (2.21)$$

where

$$\begin{aligned} P_k(n) &:= R_k\left(E_n^D - \frac{1}{2}\alpha_0\beta_0; \alpha_0 - 1, \beta_0 - 1, K\right) = \\ &= \sqrt{\binom{\alpha_0 - 1 + k}{k} \binom{\beta_0 + K - k}{K - k}} {}_3F_2\left(\begin{matrix} -k, -n, n + \alpha_0 + \beta_0 - 1 \\ \alpha_0, -K \end{matrix} \middle| 1\right) \end{aligned} \quad (2.22)$$

are normalized Dual Hahn polynomials, see [KS98]. The polynomials $\{P_k\}_{k=0}^K$ form the orthonormal basis in $L^2(\mathbb{R}, d\omega)$ with the measure given by

$$\begin{aligned} d\omega(x) &= \sum_{n=0}^K \frac{(2n + \alpha_0 + \beta_0 - 1)(\alpha_0)_n(-K)_n K!}{(-1)^n(n + \alpha_0 + \beta_0 - 1)_{K+1}(\beta_0)_n n!} \times \\ &\quad \times \delta\left(x - E_n^D - \frac{1}{2}\alpha_0\beta_0\right) dx. \end{aligned} \quad (2.23)$$

Finally we get the following spectral decomposition

$$\mathbf{H}_D = \sum_{r_0=0}^{l_0-1} \sum_{r_1=0}^{l_1-1} \sum_{K=0}^{\infty} \sum_{k=0}^K E_n^D \frac{|E_n^D\rangle\langle E_n^D|}{\langle E_n^D|E_n^D\rangle}. \quad (2.24)$$

In particular we see that the eigenspaces of \mathbf{H}_D in $\mathcal{H} \otimes \mathcal{H}$ are infinite dimensional.

2.4. Spectral decomposition of \mathbf{H}_C .

In this case the operator \mathbf{D}_0 assumes the form

$$\mathbf{D}_0 = \mathbf{A}_0 - \mathbf{B}_0. \quad (2.25)$$

Thus its spectrum consists of numbers $2K + \alpha_0 - \beta_0$ for $K \in \mathbb{Z}$ and corresponding eigenspaces

$$\mathcal{H}_{r_0, r_1}^K := \text{span}\{|k\rangle_{r_0, r_1}^K \mid k = 0, 1, \dots\} \quad (2.26)$$

are infinite dimensional, where

$$|k\rangle_{r_0, r_1}^K := \begin{cases} |K+k, k\rangle_{r_0, r_1} & \text{for } K \geq 0 \\ |k, k-K\rangle_{r_0, r_1} & \text{for } K < 0 \end{cases} . \quad (2.27)$$

So we get the decomposition

$$\mathcal{H}_{r_0, r_1} = \bigoplus_{K \in \mathbb{Z}} \mathcal{H}_{r_0, r_1}^K \quad (2.28)$$

onto subspaces invariant with respect to \mathbf{H}_C . The Hamiltonian $\mathbf{H}_C|_{\mathcal{H}_{r_0, r_1}^K}$ also acts in basis $\{|k\rangle_{r_0, r_1}^K\}_{k=0}^\infty$ like in previous section, i.e.:

$$\mathbf{H}_C |k\rangle_{r_0, r_1}^K = b_{k-1} |k-1\rangle_{r_0, r_1}^K + a_k |k\rangle_{r_0, r_1}^K + b_k |k+1\rangle_{r_0, r_1}^K , \quad (2.29)$$

where now

$$a_k = \begin{cases} -\frac{1}{2}(2(K+k) + \alpha_0)(2k + \beta_0) & \text{for } K \geq 0 \\ -\frac{1}{2}(2k + \alpha_0)(2(k-K) + \beta_0) & \text{for } K < 0 \end{cases} , \quad (2.30)$$

$$b_k = \begin{cases} -\sqrt{(K+k+\alpha_0)(K+k+1)(k+\beta_0)(k+1)} & \text{for } K \geq 0 \\ -\sqrt{(k+\alpha_0)(k+1)(k-K+\beta_0)(k-K+1)} & \text{for } K < 0 \end{cases} . \quad (2.31)$$

Three-term recurrence relation (1.27) with these coefficients is solved by normalized Continuous Dual Hahn polynomials

$$P_n(x) = S_n(-x; u, v, w) := \frac{(u+v)_n(u+w)_n}{\sqrt{\Gamma(n+u+v)\Gamma(n+u+w)\Gamma(n+v+w)n!}} {}_3F_2 \left(\begin{matrix} -n & u+\sqrt{x} & u-\sqrt{x} \\ u+v & u+w \end{matrix} \middle| 1 \right) , \quad (2.32)$$

which are orthogonal with respect to the measure

$$d\omega(x) = \left| \frac{\Gamma(u+i\sqrt{-x})\Gamma(v+i\sqrt{-x})\Gamma(w+i\sqrt{-x})}{\Gamma(2i\sqrt{-x})} \right|^2 \frac{1}{2\sqrt{-x}} \theta(-x) dx + \frac{\Gamma(u+v)\Gamma(u+w)\Gamma(v-u)\Gamma(w-u)}{\Gamma(-2a)} \times \sum_{\substack{n=0,1,2,\dots \\ u+n < 0}} \frac{(2u)_n(u+1)_n(u+v)_n(u+w)_n}{(u)_n(u-v+1)_n(u-w+1)_nn!} (-1)^n \delta(x - (u+n)^2) dx, \quad (2.33)$$

see [KS98]. The parameters u, v, w depend on the constants α_0, β_0 and K in the following way. For $K \geq 0$ we have

$$u = \begin{cases} \frac{1}{2}(\beta_0 - \alpha_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-\infty, -1) \\ \frac{1}{2}(\alpha_0 + \beta_0 - 1) & \text{if } \beta_0 - \alpha_0 \in (-1, 2K+1) \\ K + \frac{1}{2}(\alpha_0 - \beta_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (2K+1, \infty) \end{cases}$$

$$v = \begin{cases} K + \frac{1}{2}(\alpha_0 - \beta_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-\infty, -1) \\ \frac{1}{2}(\beta_0 - \alpha_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-1, 2K + 1) \\ \frac{1}{2}(\alpha_0 + \beta_0 - 1) & \text{if } \beta_0 - \alpha_0 \in (2K + 1, \infty) \end{cases}$$

$$w = \begin{cases} \frac{1}{2}(\alpha_0 + \beta_0 - 1) & \text{if } \beta_0 - \alpha_0 \in (-\infty, -1) \\ K + \frac{1}{2}(\alpha_0 - \beta_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-1, 2K + 1) \\ \frac{1}{2}(\beta_0 - \alpha_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (2K + 1, \infty) \end{cases}$$

and for $K < 0$ we have

$$u = \begin{cases} -K + \frac{1}{2}(\beta_0 - \alpha_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-\infty, -1 + 2K) \\ \frac{1}{2}(\alpha_0 + \beta_0 - 1) & \text{if } \beta_0 - \alpha_0 \in (-1 + 2K, 1) \\ \frac{1}{2}(\alpha_0 - \beta_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (1, \infty) \end{cases}$$

$$v = \begin{cases} \frac{1}{2}(\alpha_0 - \beta_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-\infty, -1 + 2K) \\ -K + \frac{1}{2}(\beta_0 - \alpha_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-1 + 2K, 1) \\ \frac{1}{2}(\alpha_0 + \beta_0 - 1) & \text{if } \beta_0 - \alpha_0 \in (1, \infty) \end{cases}$$

$$w = \begin{cases} \frac{1}{2}(\alpha_0 + \beta_0 - 1) & \text{if } \beta_0 - \alpha_0 \in (-\infty, -1 + 2K) \\ \frac{1}{2}(\alpha_0 - \beta_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (-1 + 2K, 1) \\ -K + \frac{1}{2}(\beta_0 - \alpha_0 + 1) & \text{if } \beta_0 - \alpha_0 \in (1, \infty). \end{cases}$$

Thus $\mathbf{H}_C|_{\mathcal{H}_{r_0, r_1}^K}$ admits unique self-adjoint extension (see [AG93]), so \mathbf{H}_C also admits unique self-adjoint extension in $\mathcal{H} \otimes \mathcal{H}$. The operator $F \circ \mathbf{H}_C|_{\mathcal{H}_{r_0, r_1}^K} \circ F^{-1}$, where F is defined like in (1.41), acts in $L^2(\mathbb{R}, d\omega)$ as operator of multiplication by $x - \frac{1}{4}((\alpha_0 - 1)^2 + (\beta_0 - 1)^2 - 1)$ and therefore the spectrum of $\mathbf{H}_C|_{\mathcal{H}_{r_0, r_1}^K}$ is

$$\text{spec } \mathbf{H}_C|_{\mathcal{H}_{r_0, r_1}^K} = \left(-\infty, -\frac{1}{4}((\alpha_0 - 1)^2 + (\beta_0 - 1)^2 - 1) \right) \cup \quad (2.34)$$

$$\cup \left\{ (u + n)^2 - \frac{1}{4}((\alpha_0 - 1)^2 + (\beta_0 - 1)^2 - 1) \mid n = 0, 1, \dots \wedge u + n < 0 \right\}$$

so it always consists of the continuous part (unbounded from below) and if $u < 0$ there are $-[u]$ points in the discrete part. The eigenvectors corresponding to discrete part are of the form

$$|E_n^C\rangle_{r_0, r_1}^K = \sum_{k=0}^{\infty} P_k((u + n)^2) |k\rangle_{r_0, r_1}^K \quad (2.35)$$

for $n = 0, 1, \dots, -[u]$.

We see that also in this case the eigenspaces of \mathbf{H}_C in $\mathcal{H} \otimes \mathcal{H}$ (if they exist) are infinite dimensional.

Similarly to the one-mode case one has the coherent state representation for the Lie algebra $\mathcal{A} \oplus \mathcal{B}$ and thus also for the Hamiltonian (2.9). This representation is obtained as the tensor product of the one-mode coherent state representations. Hence all formulae concerning

the two-mode coherent state representation are easily obtained by the tensoring procedure. Therefore we will not present it here.

3. PHYSICAL REMARKS

In the previous chapters we have found spectral decomposition of the Hamiltonians $\mathbf{H}_{\mu\nu}$ (1.38) and \mathbf{H} (2.9), i.e. we have integrated the considered systems.

The aim of this chapter is to discuss the physical interpretation of the Hamiltonians $\mathbf{H}_{\mu\nu}$ and \mathbf{H} . First of all let us state that we will interpret these operators as Hamiltonians of one-mode and two-mode bosonic quantum fields in non-linear medium.

We consider \mathbf{H} (resp. $\mathbf{H}_{\mu\nu}$) as interaction Hamiltonians for the system and, as shown in the paper [HOT02], we introduce full Hamiltonian by the formula

$$\mathbf{H}_F := \mathbf{H}_0 + e^{-i\mathbf{H}_0 t} \mathbf{H} e^{i\mathbf{H}_0 t}, \quad (3.1)$$

where $\mathbf{H}_0 := \omega_0 \mathbf{n}_0 + \omega_1 \mathbf{n}_1$ (resp. $\mathbf{H}_0 := \omega \mathbf{n}$) is free bosonic Hamiltonian. The solutions of Schrödinger equation are given by

$$|\psi(t)\rangle = e^{-i\mathbf{H}_0 t} e^{-i\mathbf{H} t} |\psi(0)\rangle \quad (3.2)$$

and allow us to compute the time evolution of the physical characteristics of the system, e.g. mean numbers of particles, their dispersion, Fano factors, correlation functions, and squeezing factors.

To give more explicit interpretation of \mathbf{H} let us observe that from the formulae (2.10) and (2.3), (2.4) it follows that \mathbf{H} is of the form:

$$\begin{aligned} \mathbf{H} = & g_{00}(\mathbf{n}_0, \mathbf{n}_1) + g_{+-}(\mathbf{n}_0, \mathbf{n}_1) (\mathbf{a}_0^*)^{l_0} \mathbf{a}_1^{l_1} + g_{-0}(\mathbf{n}_0, \mathbf{n}_1) \mathbf{a}_0^{l_0} + \\ & + g_{0-}(\mathbf{n}_0, \mathbf{n}_1) \mathbf{a}_1^{l_1} + g_{--}(\mathbf{n}_0, \mathbf{n}_1) \mathbf{a}_0^{l_0} \mathbf{a}_1^{l_1} + h.c.. \end{aligned} \quad (3.3)$$

The term $g_{00}(\mathbf{n}_0, \mathbf{n}_1)$ corresponds to Kerr-type effects in the non-linear medium (e.g. bistability), see [PL98]. The term $g_{+-}(\mathbf{n}_0, \mathbf{n}_1) (\mathbf{a}_0^*)^{l_0} \mathbf{a}_1^{l_1}$ describes intensity dependent (parametric) conversion of the cluster of l_0 bosons in the first mode into the cluster of l_1 bosons in the second mode, i.e. the factor $(\mathbf{a}_0^*)^{l_0} \mathbf{a}_1^{l_1}$ describes the absorption by the medium of one bosonic cluster and emission of another cluster, while the factor $g_{+-}(\mathbf{n}_0, \mathbf{n}_1)$ is intensity dependent coupling constant. The conjugated term describes the reverse conversion of clusters. By analogy the remaining terms describe the process of intensity dependent absorption of the cluster of bosons in one or both modes. The conjugated terms describe the emission of the same clusters. The process of emission-absorption occurs with the probability depending on the functions g_{kl} , where $k, l = 0, +, -$, so it essentially depends on the number of bosons in the medium.

For $ab > 0$ the spectrum of \mathbf{H} is purely discrete thus the considered system has only bound states. On the other hand for $ab < 0$ the spectrum of \mathbf{H} has continuous part and (in particular cases) discrete part. Thus the considered system has scattering states and (in some cases) bound states. Let us remark that due to the fact that in this case the spectrum is unbounded from below, it is correct from physical point of view to consider Hamiltonian $-\mathbf{H}$.

Some representatives of integrated class of Hamiltonians, related to Examples 1.1 and 1.2 are:

$$\mathbf{H}_I = \mathbf{n}_0 + \mathbf{n}_1 + 2\mathbf{n}_0 \mathbf{n}_1 + \mathbf{a}_0^2 \mathbf{a}_1^2 + (\mathbf{a}_0^*)^2 (\mathbf{a}_1^*)^2, \quad (3.4)$$

$$\mathbf{H}_{II} = \mathbf{n}_0 + \mathbf{n}_1 + 2\mathbf{n}_0 \mathbf{n}_1 + \mathbf{a}_0^2 (\mathbf{a}_1^*)^2 + \mathbf{a}_0^2 (\mathbf{a}_1^*)^2, \quad (3.5)$$

$$\mathbf{H}_{III} = \mathbf{n}_0 + \mathbf{n}_1 + 2\mathbf{n}_0 \mathbf{n}_1 + \sqrt{\mathbf{n}_0} \mathbf{a}_0^* \mathbf{a}_1^2 + \sqrt{\mathbf{n}_0 + 1} \mathbf{a}_0 (\mathbf{a}_1^*)^2, \quad (3.6)$$

$$\mathbf{H}_{IV} = \mathbf{n}_0 + \mathbf{n}_1 + 2\mathbf{n}_0 \mathbf{n}_1 + \sqrt{\mathbf{n}_0 \mathbf{n}_1} \mathbf{a}_0^* \mathbf{a}_1^* + \sqrt{(\mathbf{n}_0 + 1)(\mathbf{n}_1 + 1)} \mathbf{a}_0 \mathbf{a}_1. \quad (3.7)$$

Detailed physical analysis of systems described by Hamiltonians \mathbf{H}_{III} and \mathbf{H}_{IV} is the subject of the paper [TOH⁺06].

Physical interpretation of Hamiltonians $\mathbf{H}_{\mu\nu}$ is analogous, i.e. they describe parametric absorption-emission of one-mode of bosons in non-linear medium.

Concluding we see that investigated Hamiltonians include a wide range of non-linear processes of interaction of bosons with medium. In particular applications they can describe the models of physical systems consisting e.g. of photons, phonons, magnetons, or Cooper pairs.

APPENDIX A. SPECTRAL DECOMPOSITION OF ONE-MODE HAMILTONIANS

In this Appendix we present formulae for spectral decomposition of Hamiltonians (1.38). The problem splits into several cases.

Case 1. $\nu = 0, \mu \neq 0$

$\mathbf{H}_{\mu\nu}$ is related to Laguerre orthonormal polynomials $P_n(x) = L_n^{(\alpha_0(r)-1)}(\frac{2x}{\mu})$. In this case

$$d\omega(x) = \frac{2}{\mu} \left(\frac{2x}{\mu} \right)^{\alpha_0(r)-1} e^{-\frac{2x}{\mu}} \theta \left(\frac{x}{\mu} \right) dx \quad (A.1)$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \mathbb{R}_+ \cup \{0\}$ for $\mu > 0$ and $\text{spec}(\mathbf{H}_{\mu\nu}) = \mathbb{R}_- \cup \{0\}$ for $\mu < 0$.

Case 2. $\mu = 0, \nu \neq 0$

$\mathbf{H}_{\mu\nu}$ is also related to Laguerre orthonormal polynomials but by different formula $P_n(x) = (-1)^n L_n^{(\alpha_0(r)-1)}(\frac{2x}{\nu})$. In this case

$$d\omega(x) = \frac{2}{\nu} \left(\frac{2x}{\nu} \right)^{\alpha_0(r)-1} e^{-\frac{2x}{\nu}} \theta \left(\frac{x}{\nu} \right) dx \quad (\text{A.2})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \mathbb{R}_+ \cup \{0\}$ for $\nu > 0$ and $\text{spec}(\mathbf{H}_{\mu\nu}) = \mathbb{R}_- \cup \{0\}$ for $\nu < 0$.

Case 3. $\mu > 0, \nu < 0$

$\mathbf{H}_{\mu\nu}$ is related to Meixner-Pollaczek orthonormal polynomials $P_n(x) = P_n^{(\frac{\alpha_0(r)}{2})}(\frac{x}{2\sqrt{-\mu\nu}}; \phi)$, for $\phi = \arccos(-\frac{\mu+\nu}{\mu-\nu})$. In this case

$$d\omega(x) = \frac{1}{2\sqrt{-\mu\nu}} e^{(2\phi-\pi)\frac{x}{2\sqrt{-\mu\nu}}} \left| \Gamma \left(\frac{1}{2}\alpha_0(r) + i\frac{x}{2\sqrt{-\mu\nu}} \right) \right|^2 dx \quad (\text{A.3})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \mathbb{R}$.

Case 4. $\mu < 0, \nu > 0$

$\mathbf{H}_{\mu\nu}$ is related to Meixner-Pollaczek orthonormal polynomials $P_n(x) = P_n^{(\frac{\alpha_0(r)}{2})}(\frac{-x}{2\sqrt{-\mu\nu}}; \phi)$, for $\phi = \arccos(-\frac{\mu+\nu}{\mu-\nu})$. In this case

$$d\omega(x) = \frac{1}{2\sqrt{-\mu\nu}} e^{(2\phi-\pi)\frac{-x}{2\sqrt{-\mu\nu}}} \left| \Gamma \left(\frac{1}{2}\alpha_0(r) - i\frac{x}{2\sqrt{-\mu\nu}} \right) \right|^2 dx \quad (\text{A.4})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \mathbb{R}$.

Case 5. $\mu > \nu > 0$

$\mathbf{H}_{\mu\nu}$ is related to Meixner orthonormal polynomials $P_n(x) = M_n(\frac{x}{2\sqrt{\mu\nu}} - \frac{\alpha_0(r)}{2}; \alpha_0(r), c)$, for $c = \frac{\mu+\nu-2\sqrt{\mu\nu}}{\mu+\nu+2\sqrt{\mu\nu}}$. In this case

$$d\omega(x) = \sum_{n=0}^{\infty} \delta(x - \alpha_0(r)\sqrt{\mu\nu} - 2\sqrt{\mu\nu}n) \frac{(\alpha_0(r))_n}{n!} c^n dx \quad (\text{A.5})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \{2\sqrt{\mu\nu}n + \alpha_0(r)\sqrt{\mu\nu} \mid n = 0, 1, 2, \dots\}$. Moreover the set of vectors

$$\left\{ \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} M_k(n; \alpha_0(r), c) |k\rangle_r \right\}_{n=0}^{\infty} \quad (\text{A.6})$$

is an orthonormal basis in \mathcal{H}_r consisting of eigenvectors of $\mathbf{H}_{\mu\nu}$.

Case 6. $\mu < \nu < 0$

$\mathbf{H}_{\mu\nu}$ is related to Meixner orthonormal polynomials $P_n(x) = M_n(\frac{-x}{2\sqrt{\mu\nu}} - \frac{\alpha_0(r)}{2}; \alpha_0(r), c)$, for $c = \frac{\mu+\nu+2\sqrt{\mu\nu}}{\mu+\nu-2\sqrt{\mu\nu}}$. In this case

$$d\omega(x) = \sum_{n=0}^{\infty} \delta(x + \alpha_0(r)\sqrt{\mu\nu} + 2\sqrt{\mu\nu} n) \frac{(\alpha_0(r))_n}{n!} c^n dx \quad (\text{A.7})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \{-2\sqrt{\mu\nu} n - \alpha_0(r)\sqrt{\mu\nu} \mid n = 0, 1, 2, \dots\}$. Moreover the set of vectors

$$\left\{ \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} M_k(n; \alpha_0(r), c) |k\rangle_r \right\}_{n=0}^{\infty} \quad (\text{A.8})$$

is an orthonormal basis in \mathcal{H}_r consisting of eigenvectors of $\mathbf{H}_{\mu\nu}$.

Case 7. $\nu > \mu > 0$

$\mathbf{H}_{\mu\nu}$ is related to Meixner orthonormal polynomials $P_n(x) = (-1)^n M_n(\frac{x}{2\sqrt{\mu\nu}} - \frac{\alpha_0(r)}{2}; \alpha_0(r), c)$, for $c = \frac{\mu+\nu-2\sqrt{\mu\nu}}{\mu+\nu+2\sqrt{\mu\nu}}$. In this case

$$d\omega(x) = \sum_{n=0}^{\infty} \delta(x - \alpha_0(r)\sqrt{\mu\nu} - 2\sqrt{\mu\nu} n) \frac{(\alpha_0(r))_n}{n!} c^n dx \quad (\text{A.9})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \{2\sqrt{\mu\nu} n + \alpha_0(r)\sqrt{\mu\nu} \mid n = 0, 1, 2, \dots\}$. Moreover the set of vectors

$$\left\{ \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} (-1)^k M_k(n; \alpha_0(r), c) |k\rangle_r \right\}_{n=0}^{\infty} \quad (\text{A.10})$$

is an orthonormal basis in \mathcal{H}_r consisting of eigenvectors of $\mathbf{H}_{\mu\nu}$.

Case 8. $\nu < \mu < 0$

$\mathbf{H}_{\mu\nu}$ is related to Meixner orthonormal polynomials $P_n(x) = (-1)^n M_n(\frac{-x}{2\sqrt{\mu\nu}} - \frac{\alpha_0(r)}{2}; \alpha_0(r), c)$, for $c = \frac{\mu+\nu+2\sqrt{\mu\nu}}{\mu+\nu-2\sqrt{\mu\nu}}$. In this case

$$d\omega(x) = \sum_{n=0}^{\infty} \delta(x + \alpha_0(r)\sqrt{\mu\nu} + 2\sqrt{\mu\nu} n) \frac{(\alpha_0(r))_n}{n!} c^n dx \quad (\text{A.11})$$

and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \{-2\sqrt{\mu\nu} n - \alpha_0(r)\sqrt{\mu\nu} \mid n = 0, 1, 2, \dots\}$. Moreover the set of vectors

$$\left\{ \sqrt{\frac{n!}{(\alpha_0(r))_n c^n}} \sum_{k=0}^{\infty} (-1)^k M_k(n; \alpha_0(r), c) |k\rangle_r \right\}_{n=0}^{\infty} \quad (\text{A.12})$$

is an orthonormal basis in \mathcal{H}_r consisting of eigenvectors of $\mathbf{H}_{\mu\nu}$.

Case 9. $\mu = \nu$

In this case formula (1.39) means that $\mathbf{H}_{\mu\nu}$ is diagonal in Fock basis and spectrum $\text{spec}(\mathbf{H}_{\mu\nu}) = \{2\mu n + \mu\alpha_0(r) \mid n = 0, 1, 2, \dots\}$.

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